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FINITE-DIFFERENCE METHODS FOR BOUNDARY-VALUE PROBLEMS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract—We propose a convergent finite-difference method to construct an approximate solution of the boundary-value problems with deviating arguments. The method developed takes into account the continuity limitations of the solutions of these problems. To demonstrate the effectiveness of the proposed method numerical evidence is also included.

1. INTRODUCTION

Consider the following second-order nonlinear differential equation with deviating arguments

$$x''(t) = f(t, x(t), x(g(t))), \tag{1}$$

where

$$f \in C([a, b] \times R^2, R), \quad g \in C[a, b].$$

Let

$$c = \min\{\inf_{a \leqslant t \leqslant b} g(t), a\}$$

and

$$d = \max\{\sup_{a \leqslant t \leqslant b} g(t), b\}$$

and $\phi \in C[c, a]$, $\psi \in C[b, d]$ be given functions. If $c = a$ and/or $d = b$ then, ϕ and/or ψ are interpreted as constants. We seek a function $x \in C[c, d] \cap C^{(2)}[a, b]$ which has the property that it satisfies the boundary conditions

$$x(t) = \phi(t), \quad t \in [\text{c, a}], \tag{2}$$

and

$$x(t) = \psi(t), \quad t \in [b, d], \tag{3}$$

and $x(t)$ is a solution of equation (1) in $[a, b]$.

It is well-known that the construction of the solution of equations (1)–(3) is much more difficult than for the boundary-value problems of ordinary differential equations. When equation (1) is linear then its solution is not simply some linear combination of a particular solution with a nontrivial solution of the homogeneous equation since, in general the space of linearly independent solutions of homogeneous equations is of infinite dimension. Thus, the practical shooting-type methods developed in Refs [1–6] cannot be used. However, based on a theory of differential inequalities De Nevers and Schmitt [7] have demonstrated that the shooting method can be applied provided $g(t) \leqslant t$, i.e. equation (1) is only of delay type, also see Refs [8, 9]. Another difficulty in developing any numerical procedure for equations (1)–(3) is by the fact that in general the solutions are only of class $C^{(2)}[a, b]$, e.g. see Refs [10–12]. Having these limited continuity assumptions, Reddien and Travis [13] developed projection-type methods using polynomial splines as approximating functions. In this paper we shall keep these limitations and propose a very simple finite-difference method which converges to the solution of equations (1)–(3). The proposed method extends and improves the method given in Ref. [14]. In particular it is applicable to problems with more general functional arguments than those treated in Ref. [14].

2. EXISTENCE AND UNIQUENESS

The theory of existence and uniqueness of solutions of problems even more general than equations (1)–(3) has been considered at length in recent years, e.g. see Refs [9, 12, 15–19]. In particular, in Ref. [19] it has been proved that if $f(t, u, v)$ satisfies the Lipschitz condition

$$|f(t,u,v)-f(t,\bar{u},\bar{v})| \leqslant L|u-\bar{u}|+M|v-\bar{v}| \tag{4}$$

$\forall (t,u,v),\ (t,\bar{u},\bar{v}) \in [a,b]\times R^2$, and

$$\alpha = \tfrac{1}{8}(L+M)(b-a)^2 < 1, \tag{5}$$

then the boundary-value problem [equations (1) – (3)] has a unique solution $x(t)$. It is of interest to note that condition (5) can be replaced by a noncomparable condition

$$\beta = \left(\frac{1}{\pi^2}L + \frac{1}{2\pi}M\right)(b-a)^2 < 1. \tag{6}$$

To prove this, we follow the technique used in Refs [18, 19]. Define $S = C[c, d]$ with the finite norm

$$\|x\| = \max\left\{ \sup_{c \leqslant t \leqslant a} |x(t)|,\quad \sup_{a \leqslant t \leqslant b} \frac{|x(t)|}{\sin\dfrac{\pi(t-a)}{(b-a)}},\quad \sup_{b \leqslant t \leqslant d} |x(t)| \right\}.$$

On the Banach space S we define a mapping T as follows:

$$(Tx)(t) = l(t) + \theta(t)\int_a^b G(t,s) f(s, x(s), x(g(s)))\,\mathrm{d}s, \tag{7}$$

where $G(t, s)$ is the Green's function associated with the boundary-value problem $x'' = 0$, $x(a) = x(b) = 0$; $\theta(t) = 1$ if $a \leqslant t \leqslant b$ and zero otherwise; the function $l(t)$ is

$$\begin{aligned} l(t) &= \phi(t), \quad c \leqslant t \leqslant a \\ &= \phi(a) + \frac{\psi(b)-\phi(a)}{b-a}(t-a), \quad a \leqslant t \leqslant b \\ &= \psi(t), \quad b \leqslant t \leqslant d. \end{aligned}$$

One can verify that: (i) $(Tx)(t) = \phi(t)$, $c \leqslant t \leqslant a$; (ii) $(Tx)(t) = \psi(t)$, $b \leqslant t \leqslant d$; (iii) T is a continuous operator from S into itself; and (iv) $(Tx)''(t) = f(t, x(t), x(g(t)))$, $a \leqslant t \leqslant b$. Now, to conclude the proof it suffices to show that T is contracting on S. For this, let $x, y \in S$ then, we have

$$\begin{aligned} |(Tx)(t)-(Ty)(t)| &\leqslant \int_a^b |G(t,s)|[L|x(s)-y(s)| + M|x(g(s))-y(g(s))|]\,\mathrm{d}s \\ &\leqslant \int_a^b |G(t,s)|\left[L\sin\frac{\pi(s-a)}{(b-a)} + M\right]\|x-y\|\,\mathrm{d}s \\ &= \left[\frac{1}{\pi^2}L(b-a)^2\sin\frac{\pi(t-a)}{(b-a)} + M(t-a)(b-t)\right]\|x-y\| \\ &\leqslant \left[\frac{1}{\pi^2}L(b-a)^2 + \frac{1}{2\pi}M(b-a)^2\right]\sin\frac{\pi(t-a)}{(b-a)}\|x-y\| \end{aligned} \tag{8}$$

and hence

$$\|Tx - Ty\| \leqslant \beta \|x - y\|.$$

Since $\beta < 1$ the assertion follows.

3. FINITE-DIFFERENCE METHOD

An approximate solution of the boundary-value problem of equations (1)–(3) can be obtained by the following finite-difference method. We divide the interval $[a, b]$ into $(n + 1)$ equally-spaced intervals of length $h = (b - a)/(n + 1)$. The discrete points are given by $t_i = a + ih$, $i = O(1)n + 1$. We shall determine the numbers x_i which approximate the values $x(t_i)$ of the exact solution $x(t)$ of the boundary-value problem of equations (1)–(3) at the points t_i, $i = O(1)n + 1$ by the scheme

$$\delta^2 x_i = h^2 f(t_i, x_i, y_i), \quad 1 \leqslant i \leqslant n, \tag{9}$$

$$x_0 = \phi(a) \quad \text{and} \quad x_{n+1} = \psi(b), \tag{10}$$

where

$$\begin{aligned} \delta x_i &= x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \quad \text{and} \\ y_i &= \phi[g(t_i)] \quad \text{if} \quad g(t_i) \leqslant a \\ &= \psi[g(t_i)] \quad \text{if} \quad g(t_i) \geqslant b \\ &= x_j \quad \text{if} \quad g(t_i) = t_j \\ &= \frac{1}{h}[t_{j+1} - g(t_i)]x_j + \frac{1}{h}[g(t_i) - t_j]x_{j+1} \quad \text{if} \quad t_j < g(t_i) < t_{j+1}. \end{aligned} \tag{11}$$

The above discretization of equations (1)–(3) is equivalent to the following summary difference equation:

$$x_k = \phi(a) + \frac{\psi(b) - \phi(a)}{n + 1} k + h^2 \sum_{i=1}^{n} g(k, i) f(t_i, x_i, y_i), \quad 0 \leqslant k \leqslant n + 1, \tag{12}$$

where

$$g(k, i) = -\frac{1}{n + 1} \begin{cases} (n - k + 1)i, & 0 \leqslant i \leqslant k - 1 \\ (n - i + 1)k, & k \leqslant i \leqslant n + 1. \end{cases}$$

Lemma 3.1

Let $x \in C[c, d] \cap C^{(2)}[a, b]$ and satisfy equations (2) and (3) and y_i be as in equation (11), then

$$|x(g(t_i)) - y_i| \leqslant \max\{|x(t_j) - x_j|, \ |x(t_{j+1}) - x_{j+1}|\} + \tfrac{1}{8} h^2 M_2, \tag{13}$$

where

$$M_2 = \max_{a \leqslant t \leqslant b} |x''(t)|.$$

Proof. If $g(t_i) \leqslant a$ or $g(t_i) \geqslant b$ or $g(t_i) = t_j$ then, expression (13) is obvious. Thus, we need to consider only the case $t_j < g(t_i) < t_{j+1}$. From polynomial interpolation, we know that

$$x(t) = \frac{1}{h}(t_{j+1} - t)x(t_j) + \frac{1}{h}(t - t_j)x(t_{j+1}) + \tfrac{1}{2}(t - t_j)(t - t_{j+1})x''(p_j)$$

where $x_j < p_j < x_{j+1}$. Thus, we have

$$\begin{aligned} x(g(t_i)) - y_i &= \frac{1}{h}[t_{j+1} - g(t_i)][x(t_j) - x_j] + \frac{1}{h}[g(t_i) - t_j][x(t_{j+1}) - x_{j+1}] \\ &\quad + \tfrac{1}{2}[g(t_i) - t_j][g(t_i) - t_{j+1}]x''(p_j) \end{aligned}$$

and hence

$$|x(g(t_i)) - y_i| \leqslant \frac{1}{h}\left[|t_{j-1} - g(t_i)| + |g(t_i) - t_j|\right]\max\{|x(t_j) - x_j|, \ |x(t_{j+1}) - x_{j+1}|\}$$
$$+ \tfrac{1}{2}|g(t_i) - t_j||g(t_i) - t_{j-1}|M_2$$
$$\leqslant \max\{|x(t_j) - x_j|, |x(t_{j-1}) - x_{j-1}|\} + \tfrac{1}{8}h^2 M_2.$$

Theorem 3.2

Suppose that the function $f(t, u, v)$ satisfies the Lipschitz condition (4) and inequality (5) or (6) holds. Then, the summary difference equation (12) or equivalently equations (9)–(11), the discretization of equations (1)–(3) has a unique solution.

Proof. Let B be the space of all functions defined at t_i, $i = O(1)n + 1$ with the finite norm

$$\|x\| = \sup_{0 \leqslant i \leqslant n+1} \frac{|x_i|}{\sin \dfrac{i\pi}{n+1}}.$$

On the Banach space B we define a mapping T as follows:

$$Tx_k = \phi(a) + \frac{\psi(b) - \phi(a)}{n+1}k + h^2 \sum_{i=1}^{n} g(k, i) f(t_i, x_i, y_i). \tag{14}$$

The rest of the proof is similar to that given in Section 2, except now we need the following inequalities [20, 21]:

$$h^2 \sum_{i=1}^{n} -g(k, i)\sin\frac{i\pi}{n+1} = \frac{h^2}{4\sin^2\dfrac{\pi}{2(n+1)}}\sin\frac{k\pi}{n+1} \leqslant \frac{h^2(n+1)^2}{\pi^2}\sin\frac{k\pi}{n+1} \tag{15}$$

and

$$h^2 \sum_{i=1}^{n} -g(k, i) = h^2\frac{k(n-k+1)}{2} \leqslant \frac{h^2(n+1)^2}{2\pi}\sin\frac{k\pi}{n+1}. \tag{16}$$

Theorem 3.3

Let the conditions of Theorem 3.2 be satisfied. Then, the solution $x(t)$ of equations (1)–(3) and the solution $\{x_k\}$ of equation (12) satisfy

$$|x(t_k) - x_k| \leqslant \frac{1}{1-\beta}\left[\tfrac{1}{8}h^2 M M_2 + \max_{1 \leqslant i \leqslant n}\left|\frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right|\right]\frac{1}{2\pi}(b-a)^2 \sin\frac{k\pi}{n+1},$$
$$1 \leqslant k \leqslant n. \tag{17}$$

Proof. The solution $x(t)$ of equations (1)–(3) at the point t_k can be written as

$$x(t_k) = \phi(a) + \frac{\psi(b) - \phi(a)}{n+1}k + h^2 \sum_{i=1}^{n} g(k, i)\left[f(t_i, x(t_i), x(g(t_i))) + \frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right]. \tag{18}$$

From equations (12) and (18) and Lemma 3.1, we find

$$|x(t_k) - x_k| \leqslant h^2 \sum_{i=1}^{n} -g(k, i)\left[|f(t_i, x(t_i), x(g(t_i))) - f(t_i, x_i, y_i)| + \left|\frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right|\right]$$
$$\leqslant h^2 \sum_{i=1}^{n} -g(k, i)\left[L|x(t_i) - x_i| + M|x(g(t_i)) - y_i| + \left|\frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right|\right]$$

$$\leqslant h^2 \sum_{i=1}^{n} -g(k,i)\left[L \sin\frac{i\pi}{n+1} \|x(t_k)-x_k\| \right.$$

$$\left. + M(\|x(t_k)-x_k\| + \tfrac{1}{8}h^2 M_2) + \max_{1\leqslant i\leqslant n}\left|\frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right|\right]. \tag{19}$$

Using inequalities (15) and (16) in inequality (19), we get

$$|x(t_k)-x_k| \leqslant \beta \sin\frac{k\pi}{n+1}\|x(t_k)-x_k\|$$

$$+\left[\tfrac{1}{8}h^2 MM_2 + \max_{1\leqslant i\leqslant n}\left|\frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right|\right] + \frac{1}{2\pi}(b-a)^2 \sin\frac{k\pi}{n+1}$$

and hence

$$(1-\beta)\|x(t_k)-x_k\| \leqslant \left[\tfrac{1}{8}h^2 MM_2 + \max_{1\leqslant i\leqslant n}\left|\frac{\delta^2 x(t_i)}{h^2} - x''(t_i)\right|\right]\frac{1}{2\pi}(b-a)^2$$

which easily provides inequality (17).

The estimate (17) asserts that the solution $\{x_k\}$ of equation (12) converges to $x(t)$ the solution of equations (1)–(3) as $h \to 0$. If in addition $x'''(t)$ exists and has at most countable jump discontinuities then, from estimate (17) it is easy to show that

$$|x(t_k)-x_k| = O(h), \tag{20}$$

whereas if $x \in C[c,d] \cap C^{(3)}[a,b]$ and $x^{(iv)}(t)$ exists and has at most countable jump discontinuities, then

$$|x(t_k)-x_k| = O(h^2). \tag{21}$$

Obviously, any further continuity assumptions on the solution $x(t)$ of equations (1)–(3) will not improve the order of our method.

4. CONSTRUCTION OF THE APPROXIMATE SOLUTION

One of the important characteristics of Theorem 3.2 is that the Picard iterative scheme,

$$\delta^2 x_i^{m+1} = h^2 f(t_i, x_i^m, y_i^m),$$

$$x_0^{m+1} = \phi(a),\ x_{n+1}^{m+1} = \psi(b)$$

and

$$x_i^0 = \phi(a) + \frac{\psi(b)-\phi(a)}{n+1} i \quad m = 0,1,\ldots, \tag{22}$$

converges to the solution $\{x_i\}$ of equations (9) and (10); also, an error estimate

$$|x_k^m - x_k| \leqslant \beta^m (1-\beta)^{-1}\|x^1 - x^0\| \sin\frac{k\pi}{n+1} \tag{23}$$

is readily available.

From scheme (22), we find

$$|x_k^1 - x_k^0| \leqslant h^2 \sum_{i=1}^{n} -g(k,i)|f(t_i, x_i^0, y_i^0)|$$

$$\leqslant h^2 \max_{0\leqslant i\leqslant n-1} |f(t_i, x_i^0, y_i^0)| \cdot \frac{1}{2\pi}(n+1)^2 \sin\frac{k\pi}{n+1}$$

and hence

$$\|x^1 - x^0\| \leqslant h^2 \max_{0\leqslant i\leqslant n-1} |f(t_i, x_i^0, y_i^0)| \frac{1}{2\pi}(n+1)^2. \tag{24}$$

Using inequality (24) in expression (23), we get

$$|x_k^m - x_k| \leqslant \beta^m (1-\beta)^{-1} \frac{1}{2\pi}(b-a)^2 \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)| \sin\frac{k\pi}{n+1}. \tag{25}$$

In our next result we shall consider the convergence of the Newton iterative method,

$$\begin{aligned} \delta^2 x_i^{m+1} &= h^2\Big[f(t_i, x_i^m, y_i^m) + (x_i^{m+1} - x_i^m)\frac{\partial f}{\partial x_i^m}(t_i, x_i^m, y_i^m) \\ &\quad + (y_i^{m+1} - y_i^m)\frac{\partial f}{\partial y_i^m}(t_i, x_i^m, y_i^m)\Big], \\ x_0^{m+1} &= \phi(a), x_{n+1}^{m+1} = \psi(b) \end{aligned} \tag{26}$$

and

$$x_i^0 = \phi(a) + \frac{\psi(b) - \phi(a)}{n+1} i \quad m = 0, 1, \ldots,$$

to the solution $\{x_i\}$ of equations (9) and (10).

Theorem 4.1

Suppose that the function $f(t, u, v)$ is continuously differentiable with respect to u and v on $[a, b] \times R^2$, and

$$\left|\frac{\partial f(t, u, v)}{\partial u}\right| \leqslant L, \quad \left|\frac{\partial f(t, u, v)}{\partial v}\right| \leqslant M.$$

Then, if $3\beta < 1$ the following hold:

(i) the sequence $\{x_i^m\}$ generated by equations (26) satisfies

$$|x_k^m - x_k^0| \leqslant (1-3\beta)^{-1} \frac{1}{2\pi}(b-a)^2 \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)| \sin\frac{k\pi}{n+1}; \tag{27}$$

(ii) the sequence $\{x_i^m\}$ converges to the unique solution $\{x_i\}$ of equations (9) and (10);

(iii) a bound on the error is given by

$$|x_k - x_k^m| \leqslant \left(\frac{2\beta}{1-\beta}\right)^m (1-3\beta)^{-1} \frac{(b-a)^2}{2\pi} \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)| \sin\frac{k\pi}{n+1}. \tag{28}$$

Proof. For $m = 0$, the inequality (27) is obviously satisfied. Thus, if inequality (27) holds for m then, it suffices to show that it is also true for $m + 1$. For this, we begin with the summation representation of equations (26):

$$\begin{aligned} x_k^{m+1} &= \phi(a) + \frac{\psi(b) - \phi(a)}{n+1} k + h^2 \sum_{i=1}^{n} g(k, i)\Big[f(t_i, x_i^m, y_i^m) \\ &\quad + (x_i^{m+1} - x_i^m)\frac{\partial f}{\partial x_i^m}(t_i, x_i^m, y_i^m) + (y_i^{m+1} - y_i^m)\frac{\partial f}{\partial y_i^m}(t_i, x_i^m, y_i^m)\Big] \end{aligned} \tag{29}$$

and hence

$$\begin{aligned} |x_k^{m+1} - x_k^0| &\leqslant h^2 \sum_{i=1}^{n} -g(k, i)[|f(t_i, x_i^m, y_i^m) - f(t_i, x_i^0, y_i^0)| \\ &\quad + |f(t_i, x_i^0, y_i^0)| + L|x_i^{m+1} - x_i^m| + M|y_i^{m+1} - y_i^m|] \\ &\leqslant h^2 \sum_{i=1}^{n} -g(k, i)[2L|x_i^m - x_i^0| + 2M|y_i^m - y_i^0| \\ &\quad + L|x_i^{m+1} - x_i^0| + M|y_i^{m+1} - y_i^0| + |f(t_i, x_i^0, y_i^0)|] \end{aligned}$$

$$\leqslant \left[2\beta \|x^m - x^0\| + \beta \|x^{m+1} - x^0\| + \frac{(b-a)^2}{2\pi} \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)| \right] \sin \frac{k\pi}{n+1}. \tag{30}$$

Using inequality (27) in expression (30), we get

$$(1-\beta)\|x^{m+1} - x^0\| \leqslant [2\beta(1-3\beta)^{-1} + 1] \frac{(b-a)^2}{2\pi} \max_{0 \leqslant i \leqslant n+1} |f(t_i x_i^0, y_i^0)|$$

$$= (1-\beta)(1-3\beta)^{-1} \frac{(b-a)^2}{2\pi} \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)|$$

and hence

$$\|x^{m+1} - x^0\| \leqslant (1-3\beta)^{-1} \frac{1}{2\pi} (b-a)^2 \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)|,$$

which implies inequality (27).

Next, from equation (29) we have

$$x_k^{m+1} - x_k^m = h^2 \sum_{i=1}^{n} g(k,i) \Big[f(t_i, x_i^m, y_i^m) - f(t_i, x_i^{m-1}, y_i^{m-1})$$
$$+ (x_i^{m+1} - x_i^m) \frac{\partial f}{\partial x_i^m}(t_i, x_i^m, y_i^m) + (y_i^{m+1} - y_i^m) \frac{\partial f}{\partial y_i^m}(t_i, x_i^m, y_i^m)$$
$$- (x_i^m - x_i^{m-1}) \frac{\partial f}{\partial x_i^{m-1}}(t_i, x_i^{m-1}, y_i^{m-1})$$
$$+ (y_i^m - y_i^{m-1}) \frac{\partial f}{\partial y_i^{m-1}}(t_i, x_i^{m-1}, y_i^{m-1}) \Big] \tag{31}$$

and hence

$$|x_k^{m+1} - x_k^m| \leqslant h^2 \sum_{i=1}^{n} -g(k,i)[2L|x_i^m - x_i^{m-1}| + 2M|y_i^m - y_i^{m-1}|$$
$$+ L|x_i^{m+1} - x_i^m| + M|y_i^{m+1} - y_i^m|].$$

Now following as earlier, we get

$$\|x^{m+1} - x^m\| \leqslant 2\beta \|x^m - x^{m-1}\| + \beta \|x^{m+1} - x^m\|$$

which also gives

$$\|x^{m+1} - x^m\| \leqslant \frac{2\beta}{1-\beta} \|x^m - x^{m-1}\|.$$

Finally, an easy induction gives

$$\|x^{m+1} - x^m\| \leqslant \left(\frac{2\beta}{1-\beta}\right)^m \|x^1 - x^0\|. \tag{32}$$

Since $3\beta < 1$, inequality (32) implies that the sequence $\{x_i^m\}$ is Cauchy and from inequality (27) it converges to $\{x_i\}$ such that

$$|x_k - x_k^0| \leqslant (1-3\beta)^{-1} \frac{1}{2\pi} (b-a)^2 \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)| \sin \frac{k\pi}{n+1}. \tag{33}$$

This $\{x_i\}$ is indeed the unique solution of equations (9) and (10) and can easily be verified.

To find the error bound (28), we use inequality (32) in the triangular inequality, to obtain

$$\|x^{m+p} - x^m\| \leqslant \left(\frac{2\beta}{1-\beta}\right)^m \left(1 - \frac{2\beta}{1-\beta}\right)^{-1} \|x_1 - x_0\| \tag{34}$$

and, now taking $p \to \infty$, to find

$$\|x - x^m\| \leqslant \left(\frac{2\beta}{1-\beta}\right)^m \left(1 - \frac{2\beta}{1-\beta}\right)^{-1} \|x_1 - x_0\|.$$

From equation (29) it is easy to get

$$\|x^1 - x^0\| \leqslant (1-\beta)^{-1} \frac{(b-a)^2}{2\pi} \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)|. \tag{35}$$

Using inequality (35) in expression (34), the inequality (28) follows.

Theorem 4.2

Let the conditions of Theorem 4.1 be satisfied. Further, let $f(t, u, v)$ be twice continuously differentiable on $[a, b] \times R^2$, and

$$\left|\frac{\partial^2 f(t,u,v)}{\partial u^2}\right| \leqslant L^2 K, \quad \left|\frac{\partial^2 f(t,u,v)}{\partial u \partial v}\right| \leqslant LMK \quad \text{and} \quad \left|\frac{\partial^2 f(t,u,v)}{\partial v^2}\right| \leqslant M^2 K.$$

Then,

$$|x_k^{m+1} - x_k^m| \leqslant \frac{1}{\delta} \Delta^{2m} \sin\frac{k\pi}{n+1} \tag{36}$$

where

$$\delta = \tfrac{1}{2} K(L+M)\left(\frac{\beta}{1-\beta}\right)$$

and

$$\Delta = \delta(1-\beta)^{-1} \frac{(b-a)^2}{2\pi} \max_{0 \leqslant i \leqslant n+1} |f(t_i, x_i^0, y_i^0)|.$$

Thus, the sequence $\{x^m\}$ generated by equations (26) converges quadratically if $\Delta < 1$.

Proof. Since $f(t, u, v)$ is assumed to be twice continuously differentiable, equation (31) can be written as

$$\begin{aligned} x_k^{m+1} - x_k^m = h^2 \sum_{i=1}^n g(k,i) \Big\{ & (x_i^{m+1} - x_i^m) \frac{\partial f}{\partial x_i^m}(t_i, x_i^m, y_i^m) \\ & + (y_i^{m+1} - y_i^m) \frac{\partial f}{\partial y_i^m}(t_i, x_i^m, y_i^m) \\ & + \frac{1}{2}\left[(x_i^m - x_i^{m-1})\frac{\partial}{\partial p_i} + (y_i^m - y_i^{m-1})\frac{\partial}{\partial q_i}\right]^2 f(t_i, p_i, q_i) \Big\}, \end{aligned}$$

where p_i lies between x_i^{m-1} and x_i^m, whereas q_i lies between y_i^{m-1} and y_i^m.

Thus, as earlier, we find

$$\begin{aligned} |x_k^{m+1} - x_k^m| &\leqslant \beta \|x^{m+1} - x^m\| \sin\frac{k\pi}{n+1} \\ &\quad + \frac{Kh^2}{2} \sum_{i=1}^n -g(k,i)[L|x_i^m - x_i^{m-1}| + M|y_i^m - y_{i-1}^{m-1}|]^2 \\ &\leqslant \beta \|x^{m+1} - x^m\| \sin\frac{k\pi}{n+1} + \frac{K}{2}\beta(L+M)\|x^m - x^{m-1}\|^2 \sin\frac{k\pi}{n+1} \end{aligned}$$

and hence

$$\begin{aligned} \|x^{m+1} - x^m\| &\leqslant \frac{K(L+M)}{2}\left(\frac{\beta}{1-\beta}\right)\|x^m - x^{m-1}\|^2 \\ &= \delta \|x^m - x^{m-1}\|^2. \end{aligned}$$

Now, an easy induction gives

$$\|x^{m+1} - x^m\| \leqslant \frac{1}{\delta} [\delta \|x^1 - x^0\|]^{2^m}. \tag{37}$$

Using inequality (35) in expression (37), we get

$$\|x^{m+1} - x^m\| \leqslant \frac{1}{\delta} \Delta^{2^m}.$$

which implies inequality (36).

5. SOME EXAMPLES

Example 5.1

From the results of Section 2, it is easy to verify that the boundary-value problem

$$\left.\begin{aligned} x''(t) &= x(|t|) + |t|(6 - t^2) \\ \text{and} \qquad & \\ x(-1) &= x(1) = 1 \end{aligned}\right\} \tag{38}$$

has a unique solution $x(t) = t^2|t|$. Since $x'''(t)$ has a jump discontinuity at $t = 0$, from equation (20) we expect that our discretization will provide only the linear convergence. In Table 1, we list the maximum errors obtained for several difference choices of h.

Table 1

n	Max error, $h = \frac{2}{3^n}$	Max error, $h = \frac{2}{5^n}$	Max error, $h = \frac{2}{2^n + 1}$	Max error, $h = \frac{2}{2^n}$
1	0.513E − 01	0.334E − 01	0.513E − 01	
2	0.190E − 01	0.678E − 02	0.334E − 01	
3	0.628E − 02	0.138E − 02	0.190E − 01	0.472E − 01
4	0.212E − 02		0.100E − 01	0.119E − 01
5	0.714E − 03		0.513E − 02	0.297E − 02
6			0.262E − 02	0.744E − 03
7			0.134E − 02	0.186E − 03
8			0.676E − 03	0.465E − 04

Example 5.2

The boundary-value problem

$$\begin{aligned} &x''(t) = x(t^2), \\ &x(0) = 0 \quad \text{and} \quad x(1) = 1 \end{aligned} \tag{39}$$

has a unique solution $x(t)$. Since $x \in C^{(4)}[0, 1]$, from equation (21) we expect quadratic convergence. In Table 2, we assume the solution $x(t)$ to be exact when $h = 1/2^8$ and give maximum errors which indeed confirm the quadratic convergence. For comparison we also give the results obtained earlier in Ref. [14], where only linear convergence is achieved.

Example 5.3

The boundary-value problem

$$\begin{aligned} &x''(t) = -\frac{1}{16} \sin x(t) - (t + 1)x(t - 1) + t, \\ &x(t) = t - \tfrac{1}{2} \quad \text{if} \quad t \leqslant 0 \quad \text{and} \quad x(2) = -\tfrac{1}{2} \end{aligned} \tag{40}$$

has been solved numerically in Refs [7, 13, 22, 23]. In Table 3, we compare our results with their best obtained values.

Table 2

h	$x(0.25)$				$x(0.5)$				$x(0.75)$			
	Present value	Max error	Jain and Agarwal [14]	Max error	Present value	Max error	Jain and Agarwal [14]	Max error	Present value	Max error	Jain and Agarwal [14]	Max error
$\frac{1}{8}$	0.231098	0.226E − 03	0.233497	0.263E − 02	0.466254	0.306E − 03	0.468818	0.287E − 02	0.716326	0.244E − 04	0.718732	0.265E − 02
$\frac{1}{16}$	0.230929	0.570E − 04	0.232172	0.130E − 02	0.466025	0.770E − 04	0.467518	0.157E − 02	0.716144	0.620E − 04	0.717390	0.131E − 02
$\frac{1}{32}$	0.230886	0.140E − 04	0.231708	0.836E − 03	0.465967	0.190E − 04	0.466980	0.103E − 02	0.716097	0.150E − 04	0.716921	0.839E − 03
$\frac{1}{64}$	0.230876	0.400E − 05	0.231333	0.461E − 03	0.465953	0.500E − 05	0.466525	0.577E − 03	0.716085	0.300E − 05	0.716544	0.462E − 03
$\frac{1}{128}$	0.230873	0.100E − 05			0.465949	0.100E − 05			0.716082	—		
$\frac{1}{256}$	0.230872	Exact			0.465948	Exact			0.716082	Exact		

Table 3

	h						Best obtained values in:			
	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	Ref. [13]	Ref. [7]	Ref. [23]	Ref. [22]
$x(0.5)$	−1.524873	−1.538885	−1.542430	−1.543320	−1.543542	−1.543598	−1.542633	−1.543053	−1.543671	−1.5436158
$x(1.0)$	−2.042571	−2.072713	−2.080336	−2.082248	−2.082726	−2.082845	−2.081627	−2.081821	−2.083078	−2.0828844
$x(1.5)$	−1.938586	−1.957158	−1.961870	−1.963052	−1.963348	−1.963422	−1.962204	−1.962343	−1.963695	−1.9634456

Example 5.4

The boundary-value problem

$$x''(t) = \left\{x\left[\frac{t}{(1+2t)^2}\right]\right\}^{(1+2t)^2},$$
$$x(0) = 1 \quad \text{and} \quad x(1) = \mathrm{e} \tag{41}$$

has a unique solution $x(t) = \mathrm{e}^t$. In Table 4, we present the maximum obtained errors for $h = 1/2^n$, $3 \leqslant n \leqslant 8$.

Table 4

	h					
	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Max error	0.618E − 04	0.556E − 04	0.334E − 04	0.123E − 04	0.296E − 05	0.746E − 06

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